

ON THE FIRST STABILITY EIGENVALUE OF SURFACES WITH CONSTANT WEIGHTED MEAN CURVATURE

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ABSTRACT. Let Σ be a compact immersed surface with constant weighted mean curvature H_f in a weighted manifold (M^3, g, f) . In this paper we obtain upper bounds for the first eigenvalue of the weighted Jacobi operator on Σ in terms of H_f and the curvature of the ambient. As consequence we obtain that there is no stable self-shrinker of the mean curvature flow.

1. INTRODUCTION

Many branches of Differential Geometry, such as, Ricci flow, mean curvature flow and optimal transportation theory, are related to Riemannian manifolds endowed of a smooth positive density function (see for instance [6, 5, 12, 13, 18, 11, 14, 17] and references therein). A theory of these spaces and its curvatures goes back to Lichnerowicz [9, 10] and more recently by Bakry and Émery [3], and it has been very active area in recent years.

We recall that a weighted manifold is a Riemannian manifold (M^3, g) endowed with a real-valued smooth function $f : M \rightarrow \mathbb{R}$ which is used as a density to measure geometric objects on M . Associated to this structure we have an important second order differential operator defined by

$$\Delta_f u = \Delta u - \langle \nabla u, \nabla f \rangle,$$

where $u \in C^\infty$. This operator is know as the Drift Laplacian.

Also, following [19] and [3], the natural generalizations of the sectional and Ricci curvatures are defined as

$$(1.1) \quad \overline{Sect}_f^{2m}(X, Y) = \overline{Sect}(X, Y) + \frac{1}{2} \left(\text{Hess } f(X, X) - \frac{(df(X))^2}{2m} \right),$$

and

$$(1.2) \quad \text{Ric}_f^{2m} = \text{Ric}_f - \frac{df \otimes df}{2m},$$

where X and Y are unit and orthogonal vectors fields tangents to M , $m > 0$, and $\text{Ric}_f = \text{Ric} + \text{Hess } f$.

Now, we introduce some objects related with the theory of isometric immersions in a weighted manifold. Let $\psi : \Sigma^2 \rightarrow M^3$ be an isometric immersion of an oriented surface Σ in M^3 and consider N an unit normal vector field globally defined. We will denote by A its second fundamental form and by H the mean curvature of Σ , that is, the trace of A .

2010 *Mathematics Subject Classification.* 53C23; 58J50.

Key words and phrases. Eigenvalue estimates, Weighted area, Surfaces.

The first author was supported by CNPq/Brazil.

We recall that the weighted mean curvature, introduced by Gromov in [7], of the immersion ψ is given by

$$H_f = H + \langle N, \nabla f \rangle.$$

Along this paper, $dv_f = e^{-f} dv$ will denote the weighted area element of the surface Σ , where dv is the area element of Σ , and $|\Sigma|$ denote the weighted area of Σ . Furthermore, we will denote by K the Gaussian curvature of Σ and \overline{Sect}_Σ is the sectional curvature of M restricted to Σ .

It is a remarkable fact that, in the variational context, immersed surfaces with constant weighted mean curvature are critical points of the weighted area functional under variations that preserves the weighted volume (see [4]). Moreover, the second variation of the weighted area gives rise the weighted Jacobi operator on Σ , see [6], which is defined by

$$(1.3) \quad J_f u = \Delta_f u + (|A|^2 + \text{Ric}_f(N, N))u,$$

for any $u \in C^\infty(\Sigma)$ and $|A|^2$ is the Hilbert-Schmidt norm of A .

In this paper, encouraged by the ideas in [15, 1, 2], we investigate some geometric aspects of immersed surfaces with constant weighted mean curvature. In particular we study problems related with the first eigenvalue of the weighted Jacobi operator on compact surfaces without boundary.

Our first result reads as follows:

Theorem 1.1. *Let (M^3, g, f) be a weighted manifold with $\text{Ric}_f^{2m} \geq 2c$ and $\overline{Sect}_\Sigma \geq c$, for some $c \in \mathbb{R}$, and consider $\psi : \Sigma^2 \rightarrow M^3$ an isometric immersion of a compact surface with constant weighted mean curvature H_f . Let λ_1 be the first eigenvalue of weighted Jacobi operator. Then,*

- (i) $\lambda_1 \leq -\frac{1}{2} \left(\frac{H_f^2}{1+m} + 4c \right)$, with equality if and only if Σ is totally umbilical in M^3 , $\text{Ric}_f^{2m} = 2c$ and $df(N) = \frac{m}{1+m} H_f$ on Σ ;
- (ii) $\lambda_1 \leq -\frac{H_f^2}{(1+2m)} - 4c - \frac{8\pi(g-1)}{|\Sigma|}$, where g denotes the genus of Σ . Furthermore, the equality occurs if and only if K is constant, $\overline{Sect}_\Sigma = c$, $\text{Ric}_f^{2m} = 2c$ and $df(N) = \frac{2m}{1+2m} H_f$ on Σ .

As a consequence of this theorem we obtain

Corollary 1.1. *Under the conditions of Theorem 1.1 we set*

$$\Lambda_1 = -\left(\frac{H_f^2}{1+2m} + 4c \right) \quad \text{and} \quad \Lambda_2 = -\frac{1}{2} \left(\frac{H_f^2}{1+m} + 4c \right).$$

Assume $H_f^2 \geq -4(1+m)(1+2m)c$. If $\Lambda_1 < \lambda_1 \leq \Lambda_2$, then $g = 0$, that is, Σ is topologically a sphere.

The second result is the following:

Theorem 1.2. *Let (M^3, g, f) be a weighted manifold with $\overline{Sect}_f^{2m} \geq c$, for some $c \in \mathbb{R}$ and $\text{Hess} f \leq \sigma \cdot g$ for some real function σ on M . Consider $\psi : \Sigma^2 \rightarrow M^3$ an isometric immersion of a compact surface with constant weighted mean curvature H_f . Then,*

- (i) $\lambda_1 \leq -\frac{1}{2} \left(\frac{H_f^2}{1+m} + 4c \right)$, with equality if and only if Σ is totally umbilical in M^3 , $\text{Ric}_f^{2m} = 2c$ and $df(N) = \frac{m}{1+m} H_f$ on Σ ;
- (ii) $\lambda_1 \leq -\frac{H_f^2}{(1+2m)} - \left(4c - \frac{\int_{\Sigma} \sigma dv_f}{|\Sigma|} \right) - \frac{8\pi(g-1)}{|\Sigma|}$, where g denotes the genus of Σ .

If the equality occurs, then $\overline{\text{Sect}}_f^{2m} = c$, $\text{Ric}_f^{2m} = 2c$, $df(N) = \frac{2m}{1+2m} H_f$, $H = \frac{1}{1+2m} H_f$, and $|A|$ is a constant on Σ . Moreover, M^3 has constant sectional curvature k and e^{-f} is the restriction of a coordinate function from the appropriate canonical embedding of \mathbb{Q}_k^3 in \mathbb{E}^4 , where \mathbb{E}^4 is \mathbb{R}^4 or \mathbb{L}^4 .

Now, we will introduced the notion of stability aiming to announce some consequences.

Definition 1. Under the above notation. We say that a surface Σ is stable if the first eigenvalue λ_1 of the Jacobi operator is nonnegative. Otherwise, we say that Σ is unstable.

Our second consequence is:

Corollary 1.2. Under the assumptions of the Theorem 1.1.

- (i) If $c > 0$, then Σ cannot be stable;
- (ii) If $c = 0$, but $H_f \neq 0$, then Σ cannot be stable;
- (iii) If $c = H_f = 0$, and the genus $g \geq 2$, then Σ cannot be stable;
- (iv) If $c = 0$ and Σ is stable, then $H_f = 0$.

For the next result we recall that a Self-shrinker of the mean curvature flow is an oriented surface $x : \Sigma \rightarrow \mathbb{R}^3$ such that

$$H = -\frac{1}{2} \langle x, N \rangle,$$

where N is an unit normal vector field on Σ . So, if we consider \mathbb{R}^3 endowed with the function $f(x) = \frac{|x|^2}{4}$, then a Self-shrinker is a f -minimal surface in the Euclidean space.

The next result is a consequence of the proof of the Theorem 1.1 and it reads as follows:

Corollary 1.3. All compact self-shrinker in the 3-dimensional Euclidean space is unstable.

More generally, the triple $(\mathbb{R}^3, \delta_{ij}, |x|^2/4)$ is known as *Gaussian space* and using this notation we have that

Corollary 1.4. All compact surface in the Gaussian space with constant weighted mean curvature is unstable.

2. PRELIMINARIES

An important result for us is the classification of weighted manifolds with constant weighted sectional curvature. The result below follows closely the one in [19], and we include the proof here for the sake of completeness.

Lemma 1. *Let (M^3, g, f) be a weighted manifold. Assume that $\overline{Sect}_f = c$, then M has constant sectional curvature k , for some $k \in \mathbb{R}$. Moreover, $c = -(m-1)k$ and if f is a non constant function, then $u = e^{-f/m}$ is the restriction of a coordinate function from the appropriate canonical embedding of a space form of curvature k , \mathbb{Q}_k^3 , in \mathbb{E}^4 , where \mathbb{E}^4 is \mathbb{R}^4 or \mathbb{L}^4 .*

Proof. Let X and Y be an unit and orthogonal vectors on M . Then, by equation (1.1), we get

$$c = \overline{Sect}(X, Y) + \frac{1}{2} \left(\text{Hess } f(X, X) - \frac{(df(X))^2}{2m} \right)$$

and

$$c = \overline{Sect}(Y, X) + \frac{1}{2} \left(\text{Hess } f(Y, Y) - \frac{(df(Y))^2}{2m} \right).$$

So, there exists a smooth function $w : M \rightarrow \mathbb{R}$ such that

$$\text{Hess } f - \frac{df \otimes df}{2m} = w \cdot g.$$

Then, letting $\{E_1, E_2, X\}$ be an orthonormal frame we have

$$2c = \sum_{i=1}^2 \overline{Sect}_f(X, E_i) = \text{Ric}(X, X) + 2w.$$

Thus, by Schur's Lemma, w is a constant function and so M has constant sectional curvature, say k . Defining the function $u = e^{-f/m}$, we have that

$$(2.1) \quad \text{Hess } u = -\frac{c-k}{m} u \cdot g.$$

So, by Lemma 1.2 in [16],

$$(2.2) \quad g = dt^2 + (u')^2 g_0,$$

where g_0 is a local metric on a surface orthogonal to ∇u (a level set of u) and u' denotes the derived of u in the direction of the gradient of u .

Computing the radial sectional curvature of the metric (2.2), we have $(c + (m-1)k)u' = 0$. Since f is non constant, we have that $c = -(m-1)k$. Moreover, as u satisfies equations (2.1) and (2.2), u is the restriction of a coordinate function from the appropriate canonical embedding of \mathbb{Q}_k^3 in \mathbb{E}^4 , where \mathbb{E}^4 is \mathbb{R}^4 or \mathbb{L}^4 . \square

Now we consider a first eigenfunction $\rho \in C^\infty(\Sigma)$ of the Jacobi operator J_f , that is, $J_f \rho = -\lambda_1 \rho$; or equivalently,

$$(2.3) \quad \Delta_f \rho = (\lambda_1 + |A|^2 + \text{Ric}_f(N, N))\rho.$$

Furthermore, λ_1 is simple and it is characterized by

$$(2.4) \quad \lambda_1 = \inf \left\{ \frac{-\int_\Sigma u J_f u dv_f}{\int_\Sigma u^2 dv_f} : u \in C^\infty(\Sigma), u \neq 0 \right\}.$$

We observe that the first eigenfunction of an elliptic second-order differential operator has a sign. Therefore, without loss of generality, we can assume that $\rho > 0$. In what follows, we emphasize that every integrations are with respect to the weighted area element.

Thus,

$$\begin{aligned}
\Delta_f \ln \rho &= \Delta \ln \rho - \langle \nabla f, \nabla \ln \rho \rangle \\
&= \operatorname{div}_\Sigma (\nabla \ln \rho) - \langle \nabla f, \rho^{-1} \nabla \rho \rangle \\
&= \operatorname{div}_\Sigma (\rho^{-1} \nabla \rho) - \rho^{-1} \langle \nabla f, \nabla \rho \rangle \\
&= \rho^{-1} \operatorname{div}_\Sigma (\nabla \rho) + \langle \nabla \rho^{-1}, \nabla \rho \rangle - \rho^{-1} \langle \nabla f, \nabla \rho \rangle \\
&= \rho^{-1} (\Delta \rho - \langle \nabla f, \nabla \rho \rangle) - \rho^{-2} |\nabla \rho|^2 \\
&= \rho^{-1} \Delta_f \rho - \rho^{-2} |\nabla \rho|^2 \\
&= -(\lambda_1 + |A|^2 + \operatorname{Ric}_f(N, N)) - \rho^{-2} |\nabla \rho|^2.
\end{aligned}$$

Integrating the equality above on Σ and using the divergence theorem we have that

$$\alpha := \int_\Sigma \rho^{-2} |\nabla \rho|^2 = - \int_\Sigma (|A|^2 + \operatorname{Ric}_f(N, N)) - \lambda_1 |\Sigma|,$$

where $\alpha \geq 0$ defines a simple invariant that is independent of the choice of ρ , because λ_1 is simple. In other words

$$\lambda_1 = -\frac{1}{|\Sigma|} \left(\alpha + \int_\Sigma (|A|^2 + \operatorname{Ric}_f(N, N)) \right).$$

Let $\{E_i\}$ be an orthonormal frame in $T\Sigma$ and $\{a_{ij}\}$ the coefficients of A in the frame, using the Gauss equation

$$K = \overline{\operatorname{Sect}}_\Sigma - \langle A(X), Y \rangle^2 + \langle A(X), X \rangle \langle A(Y), Y \rangle,$$

we have that

$$K - \overline{\operatorname{Sect}}_\Sigma = a_{11}a_{22} - a_{12}^2 = \frac{1}{2} \left((a_{11} + a_{22})^2 - \sum_{i,j=1}^2 a_{ij}^2 \right) = \frac{1}{2} (H^2 - |A|^2),$$

hence

$$(2.5) \quad |A|^2 = H^2 + 2(\overline{\operatorname{Sect}}_\Sigma - K).$$

Using the Gauss-Bonnet theorem and the definition of weighted mean curvature, $H_f = H + \langle N, \nabla f \rangle$, we obtain that

$$\lambda_1 = -\frac{1}{|\Sigma|} \left\{ \alpha - 8\pi(1 - g) + \int_\Sigma [H^2 + 2\overline{\operatorname{Sect}}_\Sigma + \operatorname{Ric}_f(N, N)] \right\},$$

that is,

$$\lambda_1 = -\frac{1}{|\Sigma|} \left\{ \alpha - 8\pi(1 - g) + \int_\Sigma (H_f - \langle N, \nabla f \rangle)^2 + \int_\Sigma [2\overline{\operatorname{Sect}}_\Sigma + \operatorname{Ric}_f(N, N)] \right\}.$$

Moreover, we know that for all $a, b \in \mathbb{R}$ and $m > -1$, it holds that

$$(2.6) \quad (a + b)^2 \geq \frac{a^2}{1+m} - \frac{b^2}{m},$$

with equality if and only if $b = -\frac{m}{1+m}a$.

Applying this inequality in the term $(H_f - \langle N, \nabla f \rangle)^2$ and using the definition of $\operatorname{Ric}_f^{2m}$, we obtain that

$$\lambda_1 \leq -\frac{1}{|\Sigma|} \left\{ \alpha + 8\pi(g - 1) + \int_\Sigma \left[\frac{H_f^2}{1+2m} - \frac{\langle N, \nabla f \rangle^2}{2m} \right] + \int_\Sigma [2\overline{\operatorname{Sect}}_\Sigma + \operatorname{Ric}_f(N, N)] \right\},$$

i.e.,

(2.7)

$$\lambda_1 \leq -\frac{H_f^2}{1+2m} - \frac{1}{|\Sigma|} \left\{ \alpha + 8\pi(g-1) + \int_{\Sigma} (\text{Ric}_f^{2m}(N, N) + 2\overline{\text{Sect}}_{\Sigma}) \right\}.$$

To finalize this section, we recall the traceless of the second fundamental form of Σ , that is, the tensor ϕ defined by $\phi = A - \frac{H}{2}I$, where I denotes the identity endomorphism on $T\Sigma$. We note that $\text{tr}(\phi) = 0$ and $|\phi|^2 = |A|^2 - \frac{H^2}{2} \geq 0$, with equality if and only if Σ is totally umbilical, where $|\phi|^2$ is the Hilbert-Schmidt norm.

In the literature, ϕ is known as the total umbilicity tensor of Σ . In terms of ϕ , the Jacobi operator is rewritten as

$$(2.8) \quad J_f u = \Delta_f u + \left(|\phi|^2 + \frac{H^2}{2} + \text{Ric}_f(N, N) \right) u.$$

We use exactly this expression in next section to obtain a estimate of the first eigenvalue of the weighted Jacobi operator.

3. MAIN RESULTS

Theorem 3.1. *Let (M^3, g, f) be a weighted manifold with $\text{Ric}_f^{2m} \geq 2c$ and $\overline{\text{Sect}} \geq c$. Consider $\psi : \Sigma^2 \rightarrow M^3$ an isometric immersion of a compact surface with constant weighted mean curvature H_f . Then,*

- (i) $\lambda_1 \leq -\frac{1}{2} \left(\frac{H_f^2}{1+m} + 4c \right)$, with equality if and only if Σ is totally umbilical in M^3 , $\text{Ric}_f^{2m} = 2c$ and $df(N) = \frac{m}{1+m} H_f$ on Σ .
- (ii) $\lambda_1 \leq -\frac{H_f^2}{(1+2m)} - 4c - \frac{8\pi(g-1)}{|\Sigma|}$, where g denotes the genus of Σ . Furthermore, the equality occurs if and only if K is a constant, $\overline{\text{Sect}}_{\Sigma} = c$, $\text{Ric}_f^{2m} = 2c$ and $df(N) = \frac{2m}{1+2m} H_f$ on Σ .

Proof. (i) Choosing the constant function $u = 1$ to be the test function in (2.4) to estimate λ_1 , and using the expression in (2.8), we obtain that

$$\begin{aligned} \lambda_1 &\leq \frac{-\int_{\Sigma} 1 J_f 1}{\int_{\Sigma} 1^2} = -\frac{1}{|\Sigma|} \left[\int_{\Sigma} |\phi|^2 + \frac{1}{2} \int_{\Sigma} H^2 + \int_{\Sigma} \text{Ric}_f(N, N) \right] \\ &= -\frac{1}{|\Sigma|} \left[\int_{\Sigma} |\phi|^2 + \frac{1}{2} \int_{\Sigma} (H_f - \langle N, \nabla f \rangle)^2 + \int_{\Sigma} \text{Ric}_f(N, N) \right] \\ &\leq -\frac{1}{|\Sigma|} \left[\int_{\Sigma} |\phi|^2 + \frac{1}{2} \int_{\Sigma} \left(\frac{H_f^2}{1+m} - \frac{\langle N, \nabla f \rangle^2}{m} \right) + \int_{\Sigma} \text{Ric}_f(N, N) \right] \\ &= -\frac{H_f^2}{2(1+m)} - \frac{1}{|\Sigma|} \left[\int_{\Sigma} |\phi|^2 + \int_{\Sigma} \text{Ric}_f^{2m}(N, N) \right] \\ &\leq -\frac{H_f^2}{2(1+m)} - 2c - \frac{1}{|\Sigma|} \int_{\Sigma} |\phi|^2 \\ &\leq -\frac{1}{2} \left(\frac{H_f^2}{1+m} + 4c \right). \end{aligned}$$

If $\lambda_1 = -\frac{1}{2} \left(\frac{H_f^2}{1+m} + 4c \right)$, then all the inequalities above becomes equalities and consequently $\phi = 0$ on Σ , $\text{Ric}_f^{2m} = 2c$ and $df(N) = \frac{m}{1+m} H_f$. Therefore, Σ is totally umbilical and the normal of Σ is a direction of minimal to curvature Ric_f^{2m} .

On the other hand, if Σ is totally umbilical, $\text{Ric}_f^{2m} = 2c$ and $df(N) = \frac{m}{1+m} H_f$, we have

$$\begin{aligned} H &= H_f - df(N) \\ &= H_f - \frac{m}{1+m} H_f \\ &= \frac{1}{1+m} H_f \end{aligned}$$

and

$$2c = \text{Ric}_f^{2m}(N, N) = \text{Ric}_f(N, N) - \frac{1}{2m} (df(N))^2.$$

So

$$\begin{aligned} \text{Ric}_f(N, N) &= 2c + \frac{1}{2m} (df(N))^2 \\ &= 2c + \frac{m}{2(1+m)^2} H_f^2. \end{aligned}$$

Hence,

$$\begin{aligned} J_f &= \Delta_f + \frac{H^2}{2} + 2c + \frac{m}{2(1+m)^2} H_f^2 \\ &= \Delta_f + \frac{1}{2(1+m)^2} H_f^2 + 2c + \frac{m}{2(1+m)^2} H_f^2 \\ &= \Delta_f + \frac{1}{2(1+m)} H_f^2 + 2c, \end{aligned}$$

and thus we conclude that

$$\lambda_1 = -\frac{1}{2} \left(\frac{H_f^2}{1+m} + 4c \right).$$

(ii) As $\text{Ric}_f^{2m} \geq 2c$, $\overline{\text{Sect}} \geq c$ and $\alpha \geq 0$, we obtain by equation (2.7) that

$$\lambda_1 \leq -\frac{H_f^2}{1+2m} - 4c - 8\pi(g-1)/|\Sigma|.$$

If the equality occurs, then $\alpha = 0$, $\overline{\text{Sect}}_\Sigma = c$, $\text{Ric}_f^{2m} = 2c$, this last means that the normal direction of Σ is a direction of minimal of the curvature Ric_f^m of M^3 . Moreover, we obtain of the equation (2.6) that

$$df(N) = \frac{2m}{1+2m} H_f.$$

In fact, $\alpha = 0$ implies that ρ is a constant and using of the equation (2.3) we have that $|A|^2$ is also a constant. Furthermore,

$$df(N) = \frac{2m}{1+2m} H_f$$

implies that the mean curvature

$$H = H_f - \frac{2m}{1+2m}H_f = \frac{1}{1+2m}H_f$$

is a constant and by equation (2.5) we have that K is a constant.

On the other hand, if K is a constant, $\overline{Sect}_\Sigma = c$, $\text{Ric}_f^{2m} = 2c$ and $df(N) = \frac{2m}{1+2m}H_f$, we have that

$$\text{Ric}_f(N, N) = 2c + \frac{2m}{(1+2m)^2}H_f^2, \quad H = \frac{1}{1+2m}H_f,$$

and so

$$\begin{aligned} J_f &= \Delta_f + |A|^2 + \text{Ric}_f(N, N) \\ &= \Delta_f + H^2 + 2(c - K) + 2c + \frac{2m}{(1+2m)^2}H_f^2 \\ &= \Delta_f + \frac{H_f^2}{(1+2m)^2} + 4c - 2K + \frac{2m}{(1+2m)^2}H_f^2 \\ &= \Delta_f + \frac{H_f^2}{1+2m} + 4c - 2K, \end{aligned}$$

and this implies that

$$\lambda_1 = -\frac{H_f^2}{1+2m} - 4c + 2K.$$

Now, using the Gauss-Bonnet theorem we conclude that

$$\lambda_1 = -\frac{H_f^2}{1+2m} - 4c - \frac{8\pi(g-1)}{|\Sigma|}.$$

□

Now considering that the ambient is a 3-dimensional simply connected space form with sectional curvature c , \mathbb{Q}_c^3 . If c is positive, we assume that all surfaces are contained in a hemisphere. In this conditions we obtain the follows result:

Corollary 3.1. *Let $\psi : \Sigma^2 \rightarrow \mathbb{Q}_c^3$ be an isometric immersion of a compact and orientable surface with constant weighted mean curvature H_f and let f be one half of the square of the extrinsic distance function. Assume that $\psi(\Sigma)$ is contained in the geodesic ball center in the origin 0 and radius \sqrt{m} . Then*

$$\begin{aligned} \text{(i)} \quad \lambda_1 &\leq -\frac{1}{2} \left(\frac{H_f^2}{1+m} + 4c \right) \\ \text{(ii)} \quad \lambda_1 &\leq -\frac{H_f^2}{(1+2m)} - 4c - \frac{8\pi(g-1)}{|\Sigma|}. \end{aligned}$$

The equalities occur if and only if $\psi(\Sigma)$ is the sphere center in the origin and radius \sqrt{m} .

Proof. We know that

$$\text{Ric}_f^{2m}(N, N) = 2c + \text{Hess } r^2(N, N) - \frac{(dr^2(N))^2}{2m}$$

and

$$\begin{aligned}\text{Hess } r^2(N, N) &= \langle \nabla_N \nabla r^2, N \rangle \\ &= 2(dr(N))^2 + 2r \text{Hess } r(N, N).\end{aligned}$$

Now, using the expression of the Hessian of the distance function in a space form, we have that

$$\text{Hess } r(N, N) = \cot_c(r)[1 - (dr(N))^2],$$

where

$$\cot_c(s) = \begin{cases} \sqrt{-c} \coth(\sqrt{-c}s) & \text{if } c < 0, \\ \frac{1}{s} & \text{if } c = 0, \\ \sqrt{c} \cot(\sqrt{c}s) & \text{if } c > 0. \end{cases}$$

So,

$$\text{Hess } r^2(N, N) = 2(dr(N))^2 + 2r \cot_c(r)[1 - (dr(N))^2].$$

Now, using that the surface is contained in the ball center in the origin and radius \sqrt{m} and $(dr(N))^2 \leq 1$, we obtain that

$$\begin{aligned}\text{Ric}_f^{2m}(N, N) &= 2c + 2(dr(N))^2 + 2r \cot_c(r)(1 - (dr(N))^2) - \frac{4r^2(dr(N))^2}{2m} \\ &\geq 2c.\end{aligned}$$

Therefore by Theorem 3.1, we conclude the inequalities enunciates. To finalize, if the equalities occurs in the inequalities, then $dr(N) = 1$ and $r^2 = m$. \square

Corollary 3.2. *Let (M^3, g, f) be a weighted manifold with $\text{Ric}_f^{2m} \geq 2c$ and $\overline{\text{Sect}} \geq c$.*

(i) *There is no compact stable surface with*

$$\frac{H_f^2}{1+2m} + 4c > 0.$$

(ii) *If Σ^2 is a compact and stable surface and $\frac{H_f^2}{1+2m} + 4c = 0$, then Σ^2 is topologically a sphere or a torus.*

(iii) *If Σ^2 is a compact and stable surface such that $\frac{H_f^2}{1+2m} + 4c < 0$, then*

$$|\Sigma| \left| \frac{H_f^2}{1+2m} + 4c \right| \geq 8\pi(g-1).$$

Proof. By definition, a surface is stable if and only if $\lambda_1 \geq 0$. Thus the item (i) follows from the Theorem 3.1 (i). For the item (ii), we used that $\frac{H_f^2}{1+2m} + 4c = 0$ and Theorem 3.1 (ii) to obtain that

$$\lambda_1 \leq -\frac{8\pi(g-1)}{|\Sigma|}.$$

Now, using the stability we conclude that $g = 0$ or 1 and thus Σ is topologically a sphere or a torus.

To finalize we using the definition of stability and the Theorem 3.1 (ii). So,

$$0 \leq \lambda_1 \leq -\frac{H_f^2}{1+2m} - 4c - \frac{8\pi(g-1)}{|\Sigma|},$$

and thus

$$|\Sigma| \left| \frac{H_f^2}{1+m} + 4c \right| \geq 8\pi(g-1).$$

□

Another consequence of the Theorem 3.1 is improves the proposition 3.2 in [8] for the case in that Σ is not necessarily f -minimal.

Corollary 3.3. *Under the same assumptions of the Theorem 3.1.*

- (i) *If $c > 0$, then Σ cannot be stable;*
- (ii) *If $c = 0$, but $H_f \neq 0$, then Σ cannot be stable;*
- (iii) *If $c = H_f = 0$, and the genus $g \geq 2$, then Σ cannot be stable;*
- (iv) *If $c = 0$ and Σ is stable, then $H_f = 0$.*

Note that the item (i) and (ii) of the Corollary 3.3 are only a rewrite of the item (i) of Corollary 3.2 and the item (iii) and (iv) are rewrite of the item (ii).

Now, we recall the generalized sectional curvature

$$\overline{Sect}_f^{2m}(X, Y) = \overline{Sect}(X, Y) + \frac{1}{2} \left(\text{Hess } f(X, X) - \frac{(df(X))^2}{2m} \right),$$

where X, Y are unit and orthogonal vectors fields on M .

Moreover,

$$\text{Ric}_f^{2m}(X, X) = \sum_{i=1}^2 \overline{Sect}_f^{2m}(X, Y_i).$$

Since

$$\text{Ric}_f^{2m}(N, N) + 2\overline{Sect}_\Sigma \geq \text{Ric}_f^{2m}(N, N) + 2\overline{Sect}_f|_\Sigma - \text{Hess } f(X, X),$$

where X is a vector field on Σ . So, if $\text{Hess } f \leq \sigma g$, we rewrite the expression (2.7) by

(3.1)

$$\lambda_1 \leq -\frac{H_f^2}{1+2m} - \frac{1}{|\Sigma|} \left\{ \alpha + 8\pi(g-1) + \int_\Sigma \left(\text{Ric}_f^{2m}(N, N) + 2\overline{Sect}_f^{2m} - \sigma \right) \right\}.$$

As a consequence, we have the following result:

Theorem 3.2. *Let (M^3, g, f) be a weighted manifold with the weighted sectional curvature satisfying $\overline{Sect}_f^{2m} \geq c$, for some $c \in \mathbb{R}$, and $\text{Hess } f \leq \sigma \cdot g$ for some real function σ on M . Consider $\psi : \Sigma^2 \rightarrow M^3$ an isometric immersion of a compact surface with constant weighted mean curvature H_f . Then,*

- (i) $\lambda_1 \leq -\frac{1}{2} \left(\frac{H_f^2}{1+m} + 4c \right)$, with equality if and only if Σ is totally umbilical in M^3 , $\text{Ric}_f^{2m} = 2c$ and $df(N) = \frac{m}{1+m} H_f$ on Σ .
- (ii) $\lambda_1 \leq -\frac{H_f^2}{(1+2m)} - \left(4c - \frac{\int_\Sigma \sigma dv_f}{|\Sigma|} \right) - \frac{8\pi(g-1)}{|\Sigma|}$, where g denotes the genus of Σ .

If the equality occurs, then $\overline{\text{Sect}}_f^{2m} = c$, $\text{Ric}_f^{2m} = 2c$, $df(N) = \frac{2m}{1+2m}H_f$, $H = \frac{1}{1+2m}H_f$, and $|A|$ is a constant on Σ . Moreover, M^3 has constant sectional curvature k and e^{-f} is the restriction of a coordinate function from the appropriate canonical embedding of \mathbb{Q}_k^3 in \mathbb{E}^4 , where \mathbb{E}^4 is \mathbb{R}^4 or \mathbb{L}^4 .

Proof. The item (i) is a consequence of Theorem 3.1 (i). To second item, we using the expression in (3.1) and our hypotheses.

Now, if equality holds, then $\alpha = 0$, $\text{Ric}_f^{2m} = 2c$ and $\overline{\text{Sect}}_f^{2m} = c$. By equality in the inequality (2.6), we obtain

$$df(N) = \frac{2m}{1+2m}H_f,$$

and so

$$H = H_f - \frac{2m}{1+2m}H_f = \frac{1}{1+2m}H_f.$$

Moreover, $\alpha = 0$ imply that ρ is constant and of the equation (2.3) we have that $|A|^2$ is also a constant.

To finish, we using the Lemma 1 to conclude that M^3 has constant sectional curvature and e^{-f} has the property enunciate in case of the equality. \square

ACKNOWLEDGEMENT

We would like to thank the M. Cavalcante for useful suggestions and comments.

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